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Short Communication

On some relationships between the eigenfrequencies of torsional vibrational systems containing lumped elements

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Abstract

This communication is concerned with torsional vibrations of an elastic bar of given length and torsional rigidity to which several discs are attached. For fixed–free and fixed–fixed cases, formulas for the sums of the squared reciprocal eigenfrequencies of the vibrational system are established.

1. Introduction

The interesting paper [1] is devoted to the plane motion of a multiple pendulum consisting of a massless inextensible string with a number of concentrated masses attached to it. Braun has shown on the basis of the Vieta's trace theorem, that, for a pendulum of given total length, the sum of the squared reciprocal eigenfrequencies of small oscillations does not depend on the distribution of the masses along the string. He has further shown for the special case of a pendulum with equal and uniformly spaced masses, this statement is related to the zeros of the Laguerre polynomials.

In a recent paper [2], Braun and his co-author considered plane oscillations of a chain consisting of an elastic spring with concentrated mass points attached to it. It is shown that the sum of squared reciprocal eigenfrequencies does not depend on the number and distribution of the attached masses. In case of a chain with a free end the sum depends only on the total mass and

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location of the center of mass. In case of a chain with both ends fixed, the sum depends additionally on the radius of inertia of the system with respect to its center of the mass.

The present note is concerned with the extension and application of the theory in Ref. [2] to torsionally vibrating elastic uniform bars. Although it is reasonable to expect that obtaining the corresponding results for the torsional counterparts is straightforward, it was not so easy. Actually, the key point in Ref. [2] is to obtain the inverse of the stiffness matrix in closed form and to formulate the so called dynamical matrix. Due to the fact that Ref. [2] is written in a very compressed style on two pages, it was only possible to get further information on the corresponding inverse via a private communication with its author.

It is worth noting that already Biezeno and Grammel [3] have considered the sum of the n squared reciprocal eigenfrequencies in the case of rotational vibrations and showed that these sums are identical if taken for a single shaft with n discs and for n shafts each with a single disc in appropriate position. Although it is acknowledged that this is different from the presentation here, there is same correlation which justifies the mention of this classical reference book.

It must be stated that expressions obtained here cannot be interpreted physically as in Ref. [2] because the notion of the center of mass is not meaningful for torsionally vibrating bar-disc systems.

Two series occurring in Ref. [2] are encountered here as well. But, unlike there, in the present study, their sums are also verified mathematically.

It is hoped that the expressions derived and written in a reader-friendly style could be of some help especially to design engineers working on eigencharacteristics of torsionally vibrating systems.

2. Theory

2.1. *Uniform, fixed–free torsional system with n discs*

The system considered first in this study is shown in Fig. 1. It consist of a uniform, fixed–free torsional bar of length L and torsional rigidity $G\bar{J}$, carrying n discs of mass moment of inertia J_i

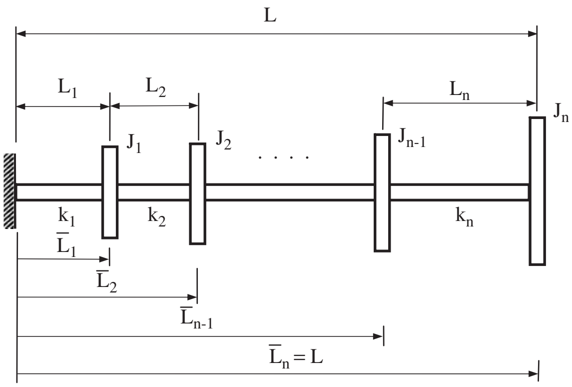

Fig. 1. Uniform, fixed–free torsional bar carrying n discs.

($i = 1, \ldots, n$). The interest here lies in establishing a relationship between the n eigenfrequencies of this torsional vibrational system.

The equivalent torsional stiffness coefficient of the bare bar is

$$k = \frac{G\bar{J}}{L}, \tag{1}$$

whereas the stiffness coefficient of the ith part of the bar can be shown to be

$$k_i = \frac{L}{L_i} k. \tag{2}$$

Equations of motion of the system can be written in the classical form

$$\mathbf{M}\ddot{\boldsymbol{\varphi}}(t) + \mathbf{K}\boldsymbol{\varphi}(t) = 0, \tag{3}$$

where the coordinate vector $\boldsymbol{\varphi}(t)$ is composed of the n torsional displacements $\varphi_i(t)$ of the n discs. The mass matrix is simply

$$\mathbf{M} = \mathbf{diag}(J_i) \quad (i = 1, \ldots, n), \tag{4}$$

whereas the stiffness matrix $\mathbf{K}$ has the form

$$\mathbf{K} = kL \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} & & & & & \\ -\frac{1}{L_2} & \frac{1}{L_2} + \frac{1}{L_3} & -\frac{1}{L_3} & & & & \\ & -\frac{1}{L_3} & \frac{1}{L_3} + \frac{1}{L_4} & -\frac{1}{L_4} & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -\frac{1}{L_{n-1}} & \frac{1}{L_{n-1}} + \frac{1}{L_n} & -\frac{1}{L_n} \\ & & & & & -\frac{1}{L_n} & \frac{1}{L_n} \end{bmatrix} \tag{5}$$

The assumption of harmonic solutions $\boldsymbol{\varphi}(t) = \bar{\boldsymbol{\varphi}} \sin \omega_{n,i} t$ leads to the standard eigenvalue problem

$$(\mathbf{A} - \lambda \mathbf{I})\bar{\boldsymbol{\varphi}} = \mathbf{0}, \tag{6}$$

where

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M}, \quad \lambda = 1/\omega_{n,i}^2 \tag{7}$$

and $\mathbf{I}$ denotes the $n \times n$ unit matrix. The matrix $\mathbf{A}$ is referred to as dynamical matrix [4].

The essential point for the developments below is that the stiffness matrix **K** can be inverted in closed form. It can be shown [5] that its inverse is

$$\mathbf{K}^{-1} = \frac{1}{kL}\begin{bmatrix} \bar{L}_1 & \bar{L}_1 & \bar{L}_1 & \cdots & \bar{L}_1 & \bar{L}_1 \\ \bar{L}_1 & \bar{L}_2 & \bar{L}_2 & \cdots & \bar{L}_2 & \bar{L}_2 \\ \bar{L}_1 & \bar{L}_2 & \bar{L}_3 & \cdots & \bar{L}_3 & \bar{L}_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \bar{L}_1 & \bar{L}_2 & \bar{L}_3 & \cdots & \bar{L}_{n-1} & \bar{L}_{n-1} \\ \bar{L}_1 & \bar{L}_2 & \bar{L}_3 & \cdots & \bar{L}_{n-1} & \bar{L}_n \end{bmatrix}, \tag{8}$$

where

$$\bar{L}_i = \sum_{k=1}^{i} L_k \quad (i = 1, \ldots, n) \tag{9}$$

are introduced, denoting aggregated lengths of various parts of the bar, measured from the fixed end.

Substitution of Eqs. (4) and (8) into Eq. (7) yields the dynamical matrix as

$$\mathbf{A} = \frac{1}{kL}\begin{bmatrix} J_1\bar{L}_1 & J_2\bar{L}_1 & J_3\bar{L}_1 & \cdots & J_{n-1}\bar{L}_1 & J_n\bar{L}_1 \\ J_1\bar{L}_1 & J_2\bar{L}_2 & J_3\bar{L}_2 & \cdots & J_{n-1}\bar{L}_2 & J_n\bar{L}_2 \\ J_1\bar{L}_1 & J_2\bar{L}_2 & J_3\bar{L}_3 & \cdots & J_{n-1}\bar{L}_3 & J_n\bar{L}_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ J_1\bar{L}_1 & J_2\bar{L}_2 & J_3\bar{L}_3 & \cdots & J_{n-1}\bar{L}_{n-1} & J_n\bar{L}_{n-1} \\ J_1\bar{L}_1 & J_2\bar{L}_2 & J_3\bar{L}_3 & \cdots & J_{n-1}\bar{L}_{n-1} & J_n\bar{L}_n \end{bmatrix}. \tag{10}$$

According to the Vieta's trace theorem [6], the trace of the matrix **A** is equal to the sum of its eigenvalues:

$$\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}\frac{1}{\omega_{n,i}^2} = \frac{1}{kL}\sum_{i=1}^{n} J_i\bar{L}_i. \tag{11}$$

Hence, Eq. (11) represents the relationship desired between the n eigenfrequencies of the system in Fig. 1.

2.2. Uniform, fixed–free torsional system with n equal discs attached at equal spacing

For the special case of a uniform n-disc bar, i.e., $L_i = L/n$, $J_i = J$ as shown in Fig. 2, the sum on the right side of Eq. (11) reduces to

$$\sum_{i=1}^{n} J_i\bar{L}_i = \frac{L}{n}J\frac{n(n+1)}{2}. \tag{12}$$

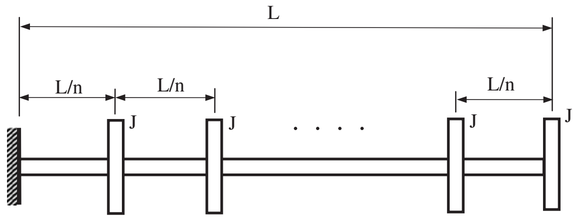

Fig. 2. Uniform, fixed–free torsional bar carrying n equal discs at equidistant spacing.

Hence, Eq. (11) yields after some rearrangements,

$$\sum_{i=1}^{n} \frac{1}{\omega_{n,i}^2} = \frac{1}{2}\left(1+\frac{1}{n}\right)\frac{IL^2}{G\bar{J}} = \frac{n(n+1)}{2}\,\frac{J}{nG\bar{J}/L}, \tag{13}$$

where $IL = nJ$ denotes the total mass moment of inertia of the discs.

By increasing the number of discs of the system to infinity in such a way that the total mass moment of inertia is kept constant, the system is turned into a homogenous torsional bar of mass moment IL and torsional stiffness $G\bar{J}$. Hence, Eq. (13) yields

$$\sum_{i=1}^{\infty} \frac{1}{\omega_{\infty,i}^2} = \frac{1}{2}\,\frac{IL^2}{G\bar{J}}, \tag{14}$$

where $\omega_{\infty,i}$ denotes the ith eigenfrequency of the system for n approaching infinity.

It is a known fact that the eigenfrequencies of uniform oscillators can be given explicitly [7]. For the uniform n-disc torsional system in Fig. 2 one obtains

$$\omega_{n,i} = 2n\,\sin\left(\frac{2i-1}{2n+1}\,\frac{\pi}{2}\right)\sqrt{\frac{G\bar{J}}{IL^2}}. \tag{15}$$

For numerical evaluations in a following section, it is suitable to rewrite the above formula also as

$$\omega_{n,i} = 2\,\sin\left(\frac{2i-1}{2n+1}\,\frac{\pi}{2}\right)\sqrt{\frac{nG\bar{J}}{JL}}, \tag{16}$$

where the expression under the square root symbol represents the squared eigenfrequency of a torsional auxiliary system composed of only one part of the bar of length L/n with a disc J at its free end.

Substitution of the eigenfrequency expression above into Eq. (13) leads to

$$\sum_{i=1}^{n} \frac{1}{\sin^2\left(\left[(2i-1)/(2n+1)\right](\pi/2)\right)} = 2n(n+1). \tag{17}$$

This result, occurring in Ref. [2] as well, which is obtained on the basis of the physical considerations can also be validated mathematically [8]. This is done in the appendix.

It is not difficult to show that expression (15) yields, for n going to infinity

$$\omega_{\infty,i} = \lim_{n\to\infty} \omega_{n,i} = \left(i - \frac{1}{2}\right)\pi\sqrt{\frac{G\bar{J}}{IL^2}} \quad (i = 1, 2, \ldots), \tag{18}$$

which is nothing else but the ith eigenfrequency of a uniform torsional bar [4]. If this expression is put into the left side of Eq. (14),

$$\sum_{i=1}^{\infty} \frac{1}{\omega_{\infty,i}^2} = \frac{4IL^2}{\pi^2 G\bar{J}} \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} \tag{19}$$

is obtained which can also be expressed as

$$\sum_{i=1}^{\infty} \frac{1}{\omega_{\infty,i}^2} = \frac{4IL^2}{\pi^2 G\bar{J}} \sum_{i=0}^{\infty} \frac{1}{(2i+1)^2}. \tag{20}$$

If, on the other side, it is considered that the sum of the series on the right side is given in handbooks [9] as $\pi^2/8$, one sees that Eq. (14) is verified.

2.3. Uniform, fixed–fixed torsional system with n discs

The torsional system to be dealt with in this section is shown in Fig. 3 which is essentially the same as in Fig. 1 except that the torsional bar is fixed at both ends. Here, in addition to L_i and $\bar{L}_i$, the aggregated lengths $\bar{\bar{L}}_i$ are defined which represent the distances of the individual discs from the right fixed end:

$$\bar{\bar{L}}_i = L - \bar{L}_i. \tag{21}$$

The mass and stiffness matrices **M** and **K** given in Eqs. (4) and (5) remain the same except that the (n, n)-element of **K** has the additive term $1/L_{n+1}$. The inverse of the stiffness matrix

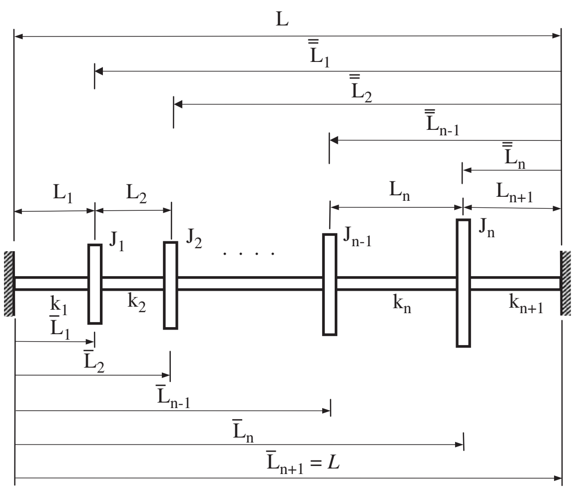

Fig. 3. Uniform, fixed–fixed torsional bar carrying n discs.

is now [5]

$$\mathbf{K}^{-1}=\frac{1}{kL^2}\begin{bmatrix} \bar{L}_1\bar{\bar{L}}_1 & \bar{L}_1\bar{\bar{L}}_2 & \bar{L}_1\bar{\bar{L}}_3 & \cdots & \bar{L}_1\bar{\bar{L}}_{n-1} & \bar{L}_1\bar{\bar{L}}_n \\ \bar{L}_1\bar{\bar{L}}_2 & \bar{L}_2\bar{\bar{L}}_2 & \bar{L}_2\bar{\bar{L}}_3 & \cdots & \bar{L}_2\bar{\bar{L}}_{n-1} & \bar{L}_2\bar{\bar{L}}_n \\ \bar{L}_1\bar{\bar{L}}_3 & \bar{L}_2\bar{\bar{L}}_3 & \bar{L}_3\bar{\bar{L}}_3 & \cdots & \bar{L}_3\bar{\bar{L}}_{n-1} & \bar{L}_3\bar{\bar{L}}_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \bar{L}_1\bar{\bar{L}}_{n-1} & \bar{L}_2\bar{\bar{L}}_{n-1} & \bar{L}_3\bar{\bar{L}}_{n-1} & \cdots & \bar{L}_{n-1}\bar{\bar{L}}_{n-1} & \bar{L}_{n-1}\bar{\bar{L}}_n \\ \bar{L}_1\bar{\bar{L}}_n & \bar{L}_2\bar{\bar{L}}_n & \bar{L}_3\bar{\bar{L}}_n & \cdots & \bar{L}_{n-1}\bar{\bar{L}}_n & \bar{L}_n\bar{\bar{L}}_n \end{bmatrix}, \tag{22}$$

which yields the dynamic matrix

$$\mathbf{A}=\frac{1}{kL^2}\begin{bmatrix} J_1\bar{L}_1\bar{\bar{L}}_1 & J_2\bar{L}_1\bar{\bar{L}}_2 & J_3\bar{L}_1\bar{\bar{L}}_3 & \cdots & J_{n-1}\bar{L}_1\bar{\bar{L}}_{n-1} & J_n\bar{L}_1\bar{\bar{L}}_n \\ J_1\bar{L}_1\bar{\bar{L}}_2 & J_2\bar{L}_2\bar{\bar{L}}_2 & J_3\bar{L}_2\bar{\bar{L}}_3 & \cdots & J_{n-1}\bar{L}_2\bar{\bar{L}}_{n-1} & J_n\bar{L}_2\bar{\bar{L}}_n \\ J_1\bar{L}_1\bar{\bar{L}}_3 & J_2\bar{L}_2\bar{\bar{L}}_3 & J_3\bar{L}_3\bar{\bar{L}}_3 & \cdots & J_{n-1}\bar{L}_3\bar{\bar{L}}_{n-1} & J_n\bar{L}_3\bar{\bar{L}}_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ J_1\bar{L}_1\bar{\bar{L}}_{n-1} & J_2\bar{L}_2\bar{\bar{L}}_{n-1} & J_3\bar{L}_3\bar{\bar{L}}_{n-1} & \cdots & J_{n-1}\bar{L}_{n-1}\bar{\bar{L}}_{n-1} & J_n\bar{L}_{n-1}\bar{\bar{L}}_n \\ J_1\bar{L}_1\bar{\bar{L}}_n & J_2\bar{L}_2\bar{\bar{L}}_n & J_3\bar{L}_3\bar{\bar{L}}_n & \cdots & J_{n-1}\bar{L}_{n-1}\bar{\bar{L}}_n & J_n\bar{L}_n\bar{\bar{L}}_n \end{bmatrix}. \tag{23}$$

The trace theorem leads now to

$$\sum_{i=1}^{n}\frac{1}{\omega_{n,i}^2}=\frac{1}{kL^2}\sum_{i=1}^{n}J_i\bar{L}_i\bar{\bar{L}}_i, \tag{24}$$

which can be brought into the following form:

$$\sum_{i=1}^{n}\frac{1}{\omega_{n,i}^2}=\frac{1}{kL}\sum_{i=1}^{n}J_i\bar{L}_i-\frac{1}{kL^2}\sum_{i=1}^{n}J_i\bar{L}_i^2. \tag{25}$$

2.4. Uniform, fixed–fixed torsional system with n equal discs attached at equal spacing

For the special case of a uniform n disc-bar torsional system as shown in Fig. 4. i.e., $L_i=L/(n+1)$, $J_i=J$, it can be shown that, after some rearrangements, Eq. (25) simplifies to:

$$\sum_{i=1}^{n}\frac{1}{\omega_{n,i}^2}=\frac{1}{6}\left(\frac{n+2}{n+1}\right)\frac{IL^2}{G\bar{J}}=\frac{n(n+2)}{6}\frac{J}{(n+1)G\bar{J}/L}. \tag{26}$$

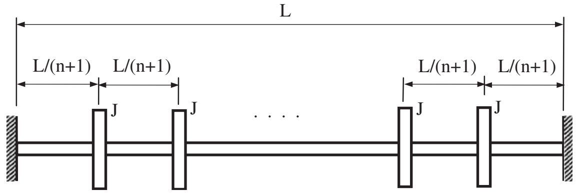

Fig. 4. Uniform, fixed–fixed torsional bar carrying n equal discs at equidistant spacing.

For the limit $n \to \infty$, Eq. (26) reduces to

$$\sum_{i=1}^{\infty} \frac{1}{\omega_{\infty,i}^2} = \frac{1}{6} \frac{IL^2}{G\bar{J}}, \tag{27}$$

which represents a property of the eigenfrequencies of a uniform torsional bar, fixed at the both ends.

The eigenfrequencies of the uniform n-disc torsional system in Fig. 4 can be given explicitly as [10]

$$\omega_{n,i} = 2\sqrt{n(n+1)} \sin\left(\frac{i}{n+1} \frac{\pi}{2}\right) \sqrt{\frac{G\bar{J}}{IL^2}}. \tag{28}$$

For numerical evaluations in the next section, it is appropriate to rewrite Eq. (28) also as

$$\omega_{n,i} = 2 \sin\left(\frac{i}{n+1} \frac{\pi}{2}\right) \sqrt{\frac{(n+1)G\bar{J}}{JL}}. \tag{29}$$

Here, the square root represents the eigenfrequency of a torsional subsystem consisting of only one part of the bar of length $L/(n+1)$ with a disc J at its free end.

Substitution of Eq. (28) into the left side of Eq. (26) gives

$$\sum_{i=1}^{n} \frac{1}{\sin^2\left([i/(n+1](\pi/2)\right)} = \frac{2}{3} n(n+2). \tag{30}$$

The mathematical validation of this interesting result [8], reported also in Ref. [2] is given in the appendix.

Expression (28) yields in the limit:

$$\omega_{\infty,i} = \lim_{n\to\infty} \omega_{n,i} = i\pi \sqrt{\frac{G\bar{J}}{IL^2}}, \tag{31}$$

which represents the ith eigenfrequency of a uniform torsional bar [4].

Substitution of the last expression into the left side of Eq. (27) results in

$$\sum_{i=1}^{\infty} \frac{1}{\omega_{\infty,i}^2} = \frac{IL^2}{\pi^2 G\bar{J}} \sum_{i=1}^{\infty} \frac{1}{i^2}. \tag{32}$$

The sum of the series on the right side of the above equation is given in handbooks [9] as $\pi^2/6$. This in turn, verifies Eq. (27) in a direct manner.

3. Numerical applications

Although the results obtained are clear and easy to apply, it is instructive to consider two simple examples, verifying numerically the expressions established.

Example 1. In the first example, $n = 5$ discs are considered in the torsional system in Fig. 2. All discs have the mass moment of inertia J and each of the five parts of the bar has the torsional stiffness coefficient $5G\bar{J}/L$ such that the whole bar has the stiffness coefficient $G\bar{J}/L$.

The nondimensionalized squares of the eigenfrequencies of the system obtained via Eq. (16) are:

0.08101405,
0.69027853,
1.71537032,
2.83083003,
3.68250707.

The sum of the inverses of these five numbers gives 15, corresponding to $5(5+1)/2$ in accordance with Eq. (13).

Example 2. The second example considers $n = 5$ discs in the torsional system given in Fig. 4. Again, all discs have the mass moment of inertia J and each of the six parts of the bar has the torsional stiffness coefficient $6G\bar{J}/L$ such that the whole bar has the stiffness coefficient $G\bar{J}/L$.

The nondimensionalized squares of the eigenfrequencies of the system obtained via Eq. (29) are:

0.26794919,
1.00000000,
2.00000000,
3.00000000,
3.73205081.

The sum of the inverses of these five numbers yields 5.83333333 which corresponds to $5(5+2)/6$ in accordance with Eq. (26).

4. Conclusions

This note is concerned with torsional vibrations of an elastic bar of given length L and torsional rigidity $G\bar{J}$ to which n discs are attached. For fixed–free and fixed–fixed cases, formulas for the sums of the squared reciprocal eigenfrequencies of the vibrational system are established. Results are also specialized to uniform torsional oscillators with equal discs at equidistant spacing. It is hoped that the expressions derived could be of some help to design engineers working on torsional vibrations of elastic bars.

Acknowledgements

The author thanks Professor Braun (Duisburg University, Germany) for his generosity in making several documents available to him and he also thanks Professor Scheufens (Technical University of Denmark) for his contributions given in the appendix.

Appendix

From Ref. [11], one has the following formula:

$$\sum_{k=0}^{(n-3)/2} \frac{1}{\sin^2\big([(2k+1)/2n]\pi\big)} = \frac{n^2}{2} + \frac{1}{4}[(-1)^n - 1]$$

for an odd integer n. The proof of the above expression can be found essentially in Ref. [12]. Substitution of

$$n = 2m + 1,$$

where m represents an arbitrary positive integer, yields

$$\sum_{k=0}^{m-1} \frac{1}{\sin^2\big([(2k+1)/(2(2m+1))]\pi\big)} = 2m(m+1).$$

This result can be arranged as

$$\sum_{k=1}^{m} \frac{1}{\sin^2\big([(2k-1)/(2m+1)](\pi/2)\big)} = 2m(m+1),$$

which corresponds to Eq. (17).

From Ref. [11], one also has the following formula:

$$\sum_{k=1}^{(n-2)/2} \frac{1}{\sin^2\big(k\pi/n\big)} = \frac{1}{6}(n^2 - 1) - \frac{1}{4}[1 + (-1)^n]$$

for an even integer n. The proof of the above formula can essentially be found in Ref. [12]. Substitution of

$$n = 2m + 2,$$

where m represents an arbitrary positive integer, yields

$$\sum_{k=1}^{m} \frac{1}{\sin^2\big(k\pi/(2m+2)\big)} = \frac{2m}{3}(m+2).$$

This result can be arranged as

$$\sum_{k=1}^{m} \frac{1}{\sin^2\big([k/(m+1)](\pi/2)\big)} = \frac{2}{3}m(m+2),$$

which corresponds to Eq. (30).

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